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Kelvin wave attenuation along nearly straight boundaries

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Two related wave problems are considered for a rotating sea of nearly uniform depth bounded by a coastline which is nearly straight. The depth changes are assumed to be independent of the distance from the coastline. The first problem, which is concerned with the origin of Kelvin waves in a coastal wave record, deals with a system of plane waves incident on the coastline and giving rise, in addition to reflected waves, to a Kelvin wave moving along the coast. Linearized theory is used to obtain details of the Kelvin wave for arbitrary perturbations in coastline and depth. Results suggest that the depth changes have their greatest effect in producing Kelvin waves if the incident wave crests are nearly parallel, but not exactly so, to the line of the depth changes. On the other hand when the wave crests are parallel to the coast, Kelvin waves are produced only by changes in the coastal boundary. In the second problem a Kelvin wave is assumed to be the incident wave. To find the energy propagated away from the coastline it is necessary to extend the theory to second order in the perturbations. It is shown that for a fixed wave period less than a pendulum day this energy has a maximum for a perturbation whose length is of comparable magnitude to the incident wavelength. Finally, the theory is applied to Kelvin waves propagating along the Californian coastline. Results obtained tend to confirm the suspicion that coastal irregularities are responsible for certain anomalies detected in tidal wave constituents by Munk, Snodgrass & Wimbush (1970).

1. Introduction

The generation of Kelvin waves appears to be mainly due to two mechanisms. One of these is atmospheric disturbance, and for a recent paper on this topic reference may be made to Thomson (1970). The other mechanism which is concerned here may be important physically, but appears to have received little attention in the literature; this is the generation of Kelvin waves at coastlines owing to the incidence of a plane wave system. Crease (1956) considers a plane wave system diffracted at a sharp edge and investigates the Kelvin wave which is thus created. In the present model the flow boundaries are assumed to be nearly straight and the Kelvin wave arises solely as a result of the energy extracted from the plane wave system by the boundary changes in coastline and depth, assumed here to be of small order ϵ. The solution of the corresponding perturbation theory is fortunately rather simple and the results are thought to be of sufficient interest to justify a brief description.

The perturbation theory leads also to a solution of what is in effect the reverse problem, i.e. the dissipation of an incident Kelvin wave at a coastline owing to boundary perturbations, and the propagation of the energy balance away from the coast in the form of cylindrical waves. This question has received some attention recently for the case when the boundary suffers an abrupt change in shape, see papers by Buchwald (1968), Packham & Williams (1968) and Packham (1969), but appears to have been overlooked for a coastline in which the changes are much less severe.

In this case the present theory must be extended to second order, $O(\epsilon^2)$, a situation which may be inferred *a priori* since the incident Kelvin wave which is $O(1)$ near the coast dies out rapidly with distance from the coastline. In this region, therefore, the fluid disturbances are at most $O(\epsilon)$, yielding an energy propagation of $O(\epsilon^2)$ which can be accounted for by a change of $O(\epsilon^2)$ in the transmitted Kelvin wave amplitude. The main object in this second problem is to reveal how the Kelvin wave reacts to small boundary perturbations placed in its path and to relate the lengths of the wave and the perturbations to the energy dissipated from the coastline.

2. Small perturbation theory

With the assumption of a time dependence of $e^{-i\omega t}$, the free-surface elevation $\zeta(x,y)$ in a rotating sea in the x, y plane satisfies the partial differential equation

$$\frac{\omega^2-f^2}{g}\zeta+H(\zeta_{xx}+\zeta_{yy})+H_x\left(\zeta_x+i\frac{f}{\omega}\zeta_y\right)=0, \tag{2.1}$$

where suffices involving variables denote differentiation with respect to these variables. Here f is the Coriolis parameter, g is the acceleration due to gravity and $H(x)$ is the depth of the sea, assumed to be a function of x only.

It may be shown that along the coastline represented by $y = L(x)$ the condition of no normal flow is

$$\zeta_x+i\frac{\omega}{f}\zeta_y = L_x\left(i\frac{\omega}{f}\zeta_x-\zeta_y\right). \tag{2.2}$$

This boundary is assumed to be almost straight, lying approximately along the x axis. The rotating sea is also assumed to be of nearly uniform depth d, occupying a region $y > 0$ in the northern hemisphere for which $f > 0$. Wave motion affected by rotation in such a sea is typified by a horizontal length scale $(gd)^{\frac{1}{2}}/f$, while the relevant vertical depth scale may be taken as the depth d. The variables x, y, ζ, L and H may therefore be conveniently replaced by their scaled counterparts $x(gd)^{\frac{1}{2}}/f$, $y(gd)^{\frac{1}{2}}/f$, $\zeta(gd)^{\frac{1}{2}}/f$, $L(gd)^{\frac{1}{2}}/f$ and dH. The above equations then take the non-dimensional form

$$k^2\zeta+H\nabla^2\zeta+H_xD'\zeta = 0 \quad \text{for} \quad y \geqslant L(x), \tag{2.3a}$$

$$D\zeta = L_xD'\zeta \quad \text{on} \quad y = L(x), \tag{2.3b}$$

where $\sigma = \omega/f$ and $k^2 = \sigma^2-1$, using the notation

$$\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \quad D = \frac{\partial}{\partial y}-\frac{i}{\sigma}\frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial x}+\frac{i}{\sigma}\frac{\partial}{\partial y}.$$

Small perturbations to both coastline and depth given by

$$L(x) = \epsilon l(x), \quad H(x) = 1+\epsilon h(x), \tag{2.4a,b}$$

where ϵ is a small quantity, are now assumed. In the analysis which follows, $l(x)$, $h(x)$ and their derivatives will be taken to be $O(1)$ in magnitude. This is equivalent to assuming that both perturbations have lengths comparable in magnitude to the horizontal scale distance $(gd)^{\frac{1}{2}}/f$, and precludes from the discussion perturbations of much smaller length.

By assuming a free-surface elevation of the form

$$\zeta = \zeta_0+\epsilon\zeta_1+\epsilon^2\zeta_2+\ldots O(\epsilon^3) \tag{2.5}$$

and using (2.4), (2.3) may be expanded in successive powers of ϵ to obtain a series of boundary-value problems:

to $O(1)$

$$\nabla^2\zeta_0+k^2\zeta_0 = 0 \quad \text{for} \quad y \geqslant 0, \tag{2.6a}$$

$$D\zeta_0 = 0 \quad \text{on} \quad y = 0; \tag{2.6b}$$

to $O(\epsilon)$

$$\nabla^2\zeta_1+k^2\zeta_1 = -h\nabla^2\zeta_0-h_xD'\zeta_0 \quad \text{for} \quad y \geqslant 0, \tag{2.7a}$$

$$D\zeta_1 = l_xD'\zeta_0-lD\zeta_{0y} \quad \text{on} \quad y = 0; \tag{2.7b}$$

to $O(\epsilon^2)$

$$\nabla^2\zeta_2+k^2\zeta_2 = -h\nabla^2\zeta_1-h_xD'\zeta_1 \quad \text{for} \quad y \geqslant 0, \tag{2.8a}$$

$$D\zeta_2 = l_xD'\zeta_1-lD\zeta_{1y}+ll_xD'\zeta_{0y}-\tfrac{1}{2}l^2D\zeta_{0yy} \quad \text{on} \quad y = 0. \tag{2.8b}$$

In addition it is necessary to satisfy the radiation condition at infinity. It is therefore assumed that (*a*) $\operatorname{Im}(\sigma) > 0$ and (*b*) ζ_1 and ζ_2 are bounded as $y \to \infty$. Eventually it is intended to let $\operatorname{Im}(\sigma) \to 0$ in order to obtain the steady wave motion desired.

The primary flow ζ_0 is taken to be a solution of (2.6), that is

$$\zeta_0 = e^{imx}\left\{e^{-iny}+e^{iny}\frac{\sigma n+im}{\sigma n-im}\right\}, \tag{2.9}$$

where $m^2+n^2 = k^2$. If m and n are real then $k^2 > 0$, and if $\operatorname{Im}(\sigma)$ is put equal to zero, as will be done eventually, (2.9) represents a system of plane waves of unit amplitude incident on and reflected by the straight boundary $y = 0$ bounding a uniform sea. The direction of propagation of the incident waves makes an angle θ with the normal to the coast, where

$$\tan\theta = m/n. \tag{2.10}$$

θ is shown in figure 1 below. The system reduces at $y = 0$ to the coastal wave

$$\zeta_0 = \frac{2\sigma n}{\sigma n-im}e^{imx}. \tag{2.11}$$

In addition to giving plane waves the primary flow (2.9) may be used to represent an incident Kelvin wave of unit amplitude travelling along the boundary in the positive x direction and given by

$$\zeta_0 = e^{i\sigma x-y}, \tag{2.12}$$

provided it is assumed that $m = \sigma$, $n = -i$ and $\operatorname{Im}(\sigma) = 0$. In this case there is no restriction on the sign of k^2.

3. Solution by using Fourier transforms

Fourier transforms are now introduced, the variable x being replaced with operator α and the transforms being denoted by an overbar, i.e.

$$\bar{\zeta}(\alpha, y) = \int_{-\infty}^{\infty} \zeta(x, y)\, e^{i\alpha x}\, dx, \tag{3.1a}$$

with inverse

$$\zeta(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\zeta}(\alpha, y)\, e^{-i\alpha x}\, d\alpha. \tag{3.1b}$$

For convergence of the Fourier transform it is assumed here that $\zeta = O(|x|^p)$, $p < 0$, as $|x| \to \infty$, and similar conditions are imposed on all other transformed quantities.

3.1. *First-order theory*: $O(\epsilon)$

Inserting the primary flow (2.9) into the first-order problem (2.7) and then transforming gives

$$\frac{d^2\bar{\zeta}_1}{dy^2} - a^2\bar{\zeta}_1 = \sum_{r=1}^{2} \bar{P}_{1,r}(\alpha)\, e^{-\delta_r y} \quad \text{for} \quad y \geqslant 0, \tag{3.2a}$$

where $a^2 = \alpha^2 - k^2$, with the boundary condition

$$i\sigma\, d\bar{\zeta}_1/dy - i\alpha\bar{\zeta}_1 = \bar{M}_1(\alpha) \quad \text{on} \quad y = 0. \tag{3.2b}$$

In (3.2) $\delta_1 = in$, $\delta_2 = -in$,

$$\bar{P}_{1,1}(\alpha) = \left\{k^2 - \left(m - \frac{ni}{\sigma}\right)(m+\alpha)\right\}\bar{h}(m+\alpha), \tag{3.3a}$$

$$\bar{P}_{1,2}(\alpha) = \frac{\sigma n + im}{\sigma n - im}\left\{k^2 - \left(m + \frac{ni}{\sigma}\right)(m+\alpha)\right\}\bar{h}(m+\alpha), \tag{3.3b}$$

$$\bar{M}_1(\alpha) = \frac{2i(\sigma^2-1)\, n}{\sigma n - im}\{m\alpha + \sigma^2\}\,\bar{l}(m+\alpha). \tag{3.3c}$$

The solution of the problem posed by (3.2), such that $|\bar{\zeta}_1|$ is bounded as $y \to \infty$, is straightforward and may be shown directly to be

$$\bar{\zeta}_1(\alpha, y) = \frac{i\bar{M}_1(\alpha)\, e^{-ay}}{\sigma a + \alpha} + \sum_{r=1}^{2} \frac{\bar{P}_{1,r}(\alpha)}{\delta_r^2 - a^2}\left\{e^{-\delta_r y} - e^{-ay}\,\frac{\alpha + \sigma\delta_r}{\alpha + \sigma a}\right\}. \tag{3.4}$$

3.2. *Second-order theory*: $O(\epsilon^2)$

In this case Fourier transforms may be applied to (2.8), and using the product rule on those terms involving ζ_1 and substituting directly for those terms in ζ_0 using (2.9) yields

$$\frac{d^2\bar{\zeta}_2}{dy^2} - a^2\bar{\zeta}_2 = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{h}(\alpha-\beta)\left\{\frac{d^2\bar{\zeta}_1}{dy^2}(\beta, y) + \frac{(\alpha-\beta)}{\sigma}\frac{d\bar{\zeta}_1}{dy}(\beta, y) - \alpha\beta\bar{\zeta}_1(\beta, y)\right\}d\beta$$

$$\text{for} \quad y \geqslant 0 \tag{3.5a}$$

and

$$i\sigma\,\frac{d^2\bar{\zeta}_2}{dy^2} - i\alpha\bar{\zeta}_2 = -\frac{i\sigma}{2\pi}\int_{-\infty}^{\infty} \bar{l}(\alpha-\beta)\left\{\frac{d^2\bar{\zeta}_1}{dy^2}(\beta, y) - \frac{\alpha}{\sigma}\frac{d\bar{\zeta}_1}{dy}(\beta, y) + \beta(\alpha-\beta)\,\bar{\zeta}_1(\beta, y)\right\}d\beta$$

$$-\frac{k^2 n(m+\alpha)}{\sigma n - im}\int_{-\infty}^{\infty} l^2(x)\, e^{i(\alpha+m)x}\, dx \quad \text{on} \quad y = 0. \tag{3.5b}$$

By substituting for $\bar{\zeta}_1(\beta, y)$ using (3.4), (3.5) may be rewritten in the form

$$\frac{d^2\bar{\zeta}_2}{dy^2} - a^2\bar{\zeta}_2 = \sum_{r=1}^{3}\int_{-\infty}^{\infty} \bar{P}_{2,r}(\alpha,\beta)\, e^{-\delta_r y}\, d\beta \quad \text{for} \quad y \geqslant 0 \tag{3.6a}$$

and

$$i\sigma\frac{d^2\bar{\zeta}_2}{dy^2} - i\alpha\bar{\zeta}_2 = \int_{-\infty}^{\infty} \bar{M}_2(\alpha,\beta)\, d\beta - \frac{k^2 n(m+\alpha)}{\sigma n - im}\int_{-\infty}^{\infty} l^2(x)\, e^{i(\alpha+m)x}\, dx \quad \text{on} \quad y = 0. \tag{3.6b}$$

In (3.6) $\delta_1 = in$, $\delta_2 = -in$ as before, $\delta_3 = b = (\beta^2 - k^2)^{\frac{1}{2}}$ and

$$\bar{P}_{2,r}(\alpha,\beta) = -\frac{1}{2\pi}\frac{\bar{h}(\alpha-\beta)}{\delta_r^2 - b^2}\left\{\delta_r^2 - \frac{\delta_r(\alpha-\beta)}{\sigma} - \alpha\beta\right\}\bar{P}_{1,r}(\beta) \quad \text{for} \quad r = 1, 2, \tag{3.7a}$$

$$\bar{P}_{2,3}(\alpha,\beta) = -\frac{1}{2\pi}\frac{\bar{h}(\alpha-\beta)}{\sigma b+\beta}\left\{b^2 - \frac{b(\alpha-\beta)}{\sigma} - \alpha\beta\right\}\left\{i\bar{M}_1(\beta) - \sum_{r=1}^{2}\frac{\sigma\delta_r+\beta}{\delta_r^2 - b^2}\bar{P}_{1,r}(\beta)\right\}, \tag{3.7b}$$

$$\bar{M}_2(\alpha,\beta) = -\frac{i\sigma}{2\pi}\left\{\frac{\bar{l}(\alpha-\beta)\, i\bar{M}_1(\beta)}{\sigma b+\beta}\left[b^2 + \frac{\alpha b}{\sigma} + \beta(\alpha-\beta)\right] + \sum_{r=1}^{2}\frac{\bar{P}_{1,r}(\beta)\,\bar{l}(\alpha-\beta)}{\delta_r^2 - b^2}\right.$$
$$\left.\times\left[\delta_r^2 + \frac{\alpha}{\sigma}\delta_r + \beta(\alpha-\beta) - \left(\frac{\sigma\delta_r+\beta}{\sigma b+\beta}\right)\left(b^2 + \frac{\alpha b}{\sigma} + \beta(\alpha-\beta)\right)\right]\right\}. \tag{3.7c}$$

The solution of (3.6) may be obtained in an identical manner to that of (3.2) above, the only difference being the extra integration with respect to β. Thus it follows that

$$\bar{\zeta}_2(\alpha,y) = \int_{-\infty}^{\infty}\frac{i\bar{M}_2(\alpha,\beta)\, e^{-ay}}{\sigma a+\alpha}\, d\beta - \frac{ik^2 n(m+\alpha)}{(\sigma n - im)(\sigma a+\alpha)}\int_{-\infty}^{\infty} l^2(x)\, e^{i(\alpha+m)x}\, dx$$
$$+ \sum_{r=1}^{3}\int_{-\infty}^{\infty}\frac{\bar{P}_{2,r}(\alpha,\beta)}{\delta_r^2 - a^2}\left\{e^{-\delta_r y} - e^{-ay}\frac{\sigma\delta_r+\alpha}{\sigma a+\alpha}\right\} d\beta. \tag{3.8}$$

Finally, (3.4) and (3.8) may be used to determine $\zeta_1(x,y)$ and $\zeta_2(x,y)$ on application of the inversion integral (3.1 *b*).

3.3. *The Kelvin wave*

The particular interest in this paper is to obtain the Kelvin wave amplitude as $x \to \infty$. Inspection of (3.4) and (3.8) reveals that this arises from the simple pole $\sigma a + \alpha = \sigma(\alpha^2 - \sigma^2 + 1)^{\frac{1}{2}} + \alpha = 0$ in the α plane, i.e. at $\alpha = -\sigma$. The residue at this pole is obtained by assuming $\mathrm{Im}(\sigma) > 0$ initially in order to satisfy the radiation condition at infinity. $\mathrm{Im}(\sigma)$ is then allowed to tend to zero to yield the required uniformly periodic flow.

For the case of incident plane waves (2.9), it can be shown that as $x \to \infty$ the first-order Kelvin wave component in the coastal wave record arising from (3.4) is

$$\zeta(x,y) = \frac{-2\sigma n i\epsilon}{\sigma n - im}\left\{(\sigma - m)\,\bar{l}(m-\sigma) + \frac{m\bar{h}(m-\sigma)}{\sigma(m+\sigma)}\right\} e^{i\sigma x - y} + O(\epsilon^2). \tag{3.9}$$

In addition to this Kelvin wave there will be a first-order reflected wave also with a non-zero amplitude at infinity. This is a direct result of the assumption of a two-dimensional depth perturbation, and stems from the simple pole below the real α axis at $\alpha = -m = -(\sigma^2 - 1 - n^2)^{\frac{1}{2}}$, where $\text{Im}(m) > 0$, for n real and $\text{Im}(\sigma) > 0$. Evaluating the residue for this contribution yields

$$\zeta(x,y) = -(ik^2/2m)\,\bar{h}(0)\,\epsilon\zeta_0(x,y), \tag{3.10}$$

where ζ_0 is the primary flow (2.9). The result (3.10) reveals that the theory becomes singular at $m = 0$. This corresponds by (2.10) to $\theta = 0$, the case when the incident wave crests are parallel to the coast and are moving along the line of the depth changes.

Finally, when the incident wave is the Kelvin wave (2.12), the results (3.4) and (3.8) may be simplified considerably by substituting $m = \sigma$, $n = -i$. After calculation of the residues at the pole $\alpha = -\sigma$, (2.5) yields the transmitted Kelvin wave as $x \to \infty$ in the form

$$\zeta(x,y) = (1 - \tfrac{1}{2}i\sigma\bar{h}(0)\,\epsilon - \tfrac{1}{8}\sigma^2\bar{h}(0)^2\,\epsilon^2 + A\epsilon^2)\,e^{i\sigma x - y} + O(\epsilon^3). \tag{3.11}$$

It may be noted that the first-order term here is not equal to that obtained from (3.9) by putting $m = \sigma$, $n = -i$. This is because the poles at $\alpha = -\sigma$ and $\alpha = -m$ now coincide. The difference is in fact supplied by the term on the right-hand side of (3.10), which reduces to a Kelvin wave in these circumstances.

The terms in (3.11) of $O(\epsilon^2)$ involve an integration with respect to β along the real axis, excluding a simple pole at $\beta = -\sigma$ and also the branch points in the case $\sigma > 1$ since these then occur at $\beta = \pm k$ on the real axis. The term in $\bar{h}^2(0)$ arises directly from the pole at $\beta = -\sigma$, the rest of the integral being given by the term in A, which will be discussed in § 4.

4. Incident plane waves

Here attention is confined to the result (3.9), some important properties of which may be conveniently illustrated by a particular example. Before doing this, however, it is of interest to point out briefly some conditions under which no Kelvin waves are produced by incident plane waves of a given period. By replacing $m - \sigma$ by α in the integrand of (3.9) it may be deduced that such a state of affairs will occur for a combination of coastal and depth changes whose transforms $\bar{l}(\alpha)$ and $\bar{h}(\alpha)$ are related by

$$\bar{l}(\alpha) = \frac{(\alpha+\sigma)}{\sigma\alpha(\alpha+2\sigma)}\,\bar{h}(\alpha). \tag{4.1}$$

It is also possible to reach a similar conclusion for the individual distributions of $l(x)$ or $h(x)$. To illustrate this consider two equal coastal perturbations l_1 a distance $2x_0$ apart and symmetrically placed with respect to the origin, i.e. take $l(x) = l_1(x - x_0) + l_1(x + x_0)$, yielding $\bar{l}(\alpha) = 2\bar{l}_1(\alpha)\cos\alpha x_0$. By using (3.9) and replacing α by $m - \sigma$ it follows that no Kelvin wave is possible for periods given by integral values of r for which

$$(m - \sigma)\,x_0 = \tfrac{1}{2}(2r + 1)\,\pi \tag{4.2}$$

and which are at the same time consistent with acute angles of incidence θ, as defined by (2.10). If on the other hand the periods are such that

$$(m-\sigma)\,x_0 = r\pi, \tag{4.3}$$

the effect of the pair of perturbations is directly additive.

Further superposition of such distributions can lead to resonance. For an example it is assumed that

$$l(x) = \sum_{s=1}^{\infty} \frac{l_1(x-(2s+1)\,x_0)+l_1(x+(2s+1)\,x_0)}{(2s+1)}, \tag{4.4}$$

limiting choice to a perturbation which tends to zero as $|x| \to \infty$. Thus it follows that

$$\bar{l}(\alpha) = 2\bar{l}_1(\alpha) \sum_{s=1}^{\infty} \frac{\cos{(2s+1)\,\alpha x_0}}{(2s+1)} = \tfrac{1}{2}\bar{l}_1(\alpha)\log\left\{\frac{1+\cos\alpha x_0}{1-\cos\alpha x_0}\right\}. \tag{4.5}$$

This again yields no Kelvin wave if x_0 is given by (4.2), and is finite for all x_0 except those given by (4.3), thus corresponding to a condition of resonance.

Consider now a particular example in some detail and assume that

$$l(x)/l(0) = h(x)/h(0) = e^{-\gamma|x|}, \tag{4.6}$$

where $l(0)$, $h(0)$ and γ are constants. Transforming then yields

$$\bar{l}(\alpha)/l(0) = \bar{h}(\alpha)/h(0) = 2\gamma/(\gamma^2+\alpha^2). \tag{4.7}$$

Choosing the coastal wave amplitude (2.11) as a standard the ratio of Kelvin wave amplitude to that of the coastal wave by (3.9) becomes

$$l(0)\,\frac{2\epsilon\tau\gamma(1-m\tau)}{\gamma^2\tau^2+(1-m\tau)^2}+h(0)\,\frac{2\epsilon\tau^4\gamma m}{(1+m\tau)\,(\gamma^2\tau^2+(1-m\tau)^2)}, \tag{4.8}$$

where $\tau = 1/\sigma$ is the wave period measured in pendulum days. In terms of non-dimensional quantities the angle of incidence (2.10) is given by

$$\tan\theta = \tau m(1-(1+m^2)\,\tau^2)^{-\frac{1}{2}}. \tag{4.9}$$

By restricting attention for the moment to the first term in (4.8) it becomes evident that the maximum amplitude ratio is $\epsilon l(0)$, when $1-m\tau = \tau\gamma$. By eliminating m it follows that the maximum amplitude ratio corresponds to an angle of incidence θ_m such that

$$\tan\,\theta_m = (1-\tau\gamma)\,[2\tau\gamma-\tau^2(1+\gamma^2)]^{-\frac{1}{2}}. \tag{4.10}$$

Figure 1 shows the variation of θ_m with τ, the period measured in pendulum days, for $\gamma = 0{\cdot}5$, $0{\cdot}8$, 1, 2 and 4. For the majority of τ and γ values it may be seen that $\theta_m > 0$. However, as γ increases, which is equivalent to the coastal perturbations $l(x)$ becoming more abrupt, θ_m becomes smaller and may even become negative for $\gamma > 1$ as τ increases. The conditions imposed in the present theory are satisfied provided that the physical length of the profile is not small compared with $f/(gd)^{\frac{1}{2}}$. If the scaled length of the profile (4.6) is taken for convenience as $4/\gamma$, i.e. twice the length in the x direction for the profile to reduce from its maximum by a factor of $1/e^2$ ($= 0{\cdot}135$), then the theory is valid provided $4/\gamma \geqslant O(1)$.

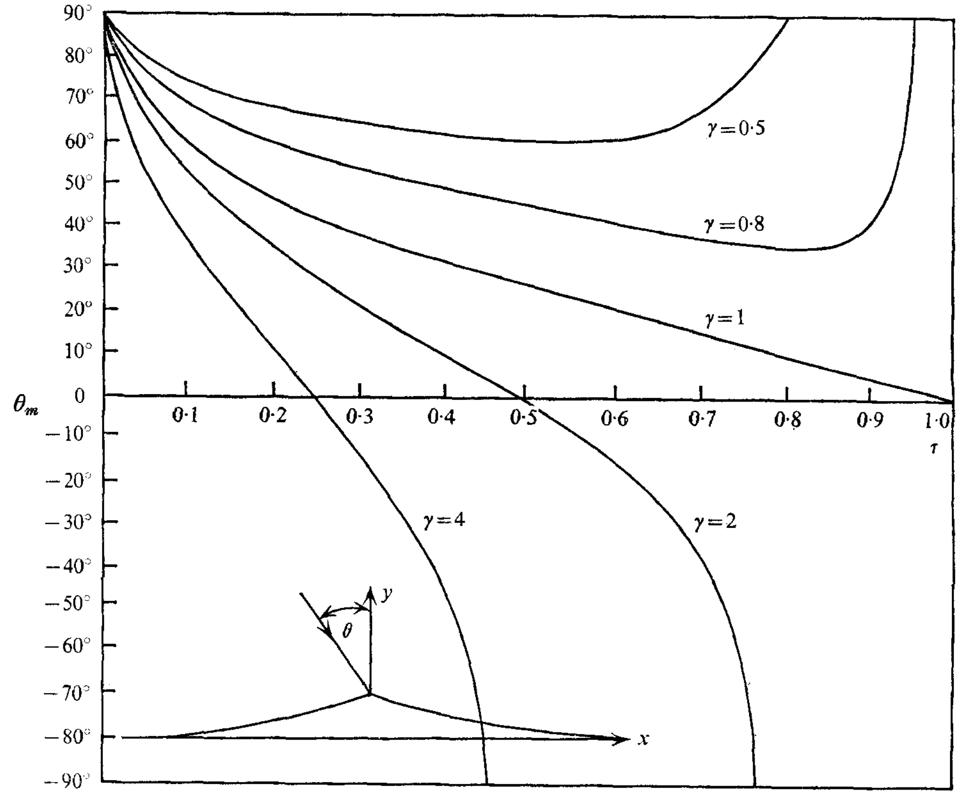

FIGURE 1. Graph of θ_m, the angle of incidence of plane waves defined by (4.10) for the maximum ratio of Kelvin wave amplitude to coastal wave amplitude.

In contrast to this situation, when the perturbation is one of depth $h(x)$ alone, inspection of the second term in (4.8) shows that no Kelvin wave is possible if $m = 0$. This corresponds by (4.6) to an incident angle $\theta = 0$ and the theory is singular, as indicated in § 3. In fact it can be shown, although the details are omitted here, that the largest values of the amplitude ratio tend to occur mostly at large incident angles, either positive or negative. From (2.10) and (2.11) it may be deduced that the coastal wave amplitude itself is zero at $\theta = \pm\frac{1}{2}\pi$, but rises rapidly for incidences away from the extremes. Thus it may be concluded that the depth changes are having greatest effect in producing Kelvin waves for wave systems whose crests are almost parallel, although not exactly so, to the line of the depth changes. Although this tendency stems from a particular case it is suspected from the form of (3.9) that it is a general result for two-dimensional depth perturbations in the presence of a coastline.

5. Incident Kelvin waves

5.1. *Amplitude changes at large distance*

Consider now the case when the incident wave is a Kelvin wave given by (2.12). It may be seen from (3.11) that although the phase of the transmitted wave is altered to $O(\epsilon)$ the amplitude reduces simply to $1+\mathrm{Re}\,(A)\,\epsilon^2+O(\epsilon^3)$, the term in $h(0)$ cancelling out, and $\mathrm{Im}\,(A)$ not contributing to this order of accuracy. Determination of that part of the residue containing $\mathrm{Re}\,(A)$ reveals that it is non-zero only in the case $\sigma > 1$, as a result of integrating along the real β axis between the branch points $\beta = \pm k$. If $1 > \sigma$, these branch points are off the real axis and no contribution of this type arises.

For $\sigma > 1$ the contribution is conveniently expressed in terms of the symmetrical components l_s and h_s and the unsymmetrical components l_u and h_u of the perturbations defined by

$$l(\pm x) = l_s(x) \pm l_u(x),$$
$$h(\pm x) = h_s(x) \pm h_u(x).$$

By putting
$$\bar{l}_s(\alpha) = \int_0^\infty l_s(x) \cos \alpha x \, dx,$$
and
$$\bar{l}_u(\alpha) = \int_0^\infty l_u(x) \sin \alpha x \, dx,$$

with similar expressions for $\bar{h}_s(\alpha)$ and $\bar{h}_u(\alpha)$, it may be shown that

$$\mathrm{Re}\,(A) = -\sum_{r=s,\,u} 2\sigma \int_{-k}^{k} (k^2-\beta^2)^{\frac{1}{2}} \frac{\sigma+\beta}{\sigma-\beta} \left\{ \bar{l}_r(\beta+\sigma) - \frac{\beta}{\sigma(\sigma^2-\beta^2)} \bar{h}_r(\beta+\sigma) \right\}^2 d\beta. \tag{5.1}$$

Since the energy is directly associated with the square of the amplitude of the Kelvin wave the present results verify to the accuracy taken that no energy is lost from the boundary in the case $1 > \sigma$ by the presence of the perturbations. If $\sigma > 1$, however, since $\mathrm{Re}\,(A) \leqslant 0$ there must always be some energy dissipation of this type except in one circumstance. This arises, it is interesting to note, when the perturbations $l(x)$ and $h(x)$ are again related by (4.1). Such combinations of coastline and depth perturbations therefore have for a given wave period the dual property that they neither extract energy from an incident plane wave system to form a Kelvin boundary wave, nor do they divert energy from an incident Kelvin wave and propagate it to infinity.†

As a convenient example the symmetric profile (4.6) is chosen to represent a coastal perturbation $l(x)$. A similar depth change is expected to lead more or less to the same results and consideration of this is therefore omitted. The ratio of the energy dissipated from the boundary to the energy of the incident wave is

$$|2\mathrm{Re}\,(A)\,\epsilon^2| = 4\epsilon^2 l^2(0)\, J(\tau,\gamma), \tag{5.2}$$

where
$$J(\tau,\gamma) = \int_{-1}^{1} (1-\alpha^2)^{\frac{1}{2}} \frac{(1+(1-\tau^2)^{\frac{1}{2}}\alpha)\,\gamma^2\tau(1-\tau^2)\,d\alpha}{[1-(1-\tau^2)^{\frac{1}{2}}\alpha]\{\gamma^2\tau^2+[1+(1-\tau^2)^{\frac{1}{2}}\alpha]^2\}^2}. \tag{5.3}$$

$J(\tau,\gamma)$ is plotted against γ in figure 2 for wave periods $\tau = 0{\cdot}2$, $0{\cdot}35$ and $0{\cdot}5$. For a given value of τ, corresponding to a Kelvin wave of length $2\pi\tau$, it may be

† A similar result in fact applies when the boundary changes are more extreme. Packham & Williams (1968) show that Kelvin waves, even for $\sigma > 1$, are refracted perfectly with no loss in energy at sharp angled bends of angle $\pi/(2r+1)$, where r is any positive integer. It is observed here that these angles are precisely those which will allow perfect reflexion by the boundaries of plane waves moving parallel to the line bisecting the vertex.

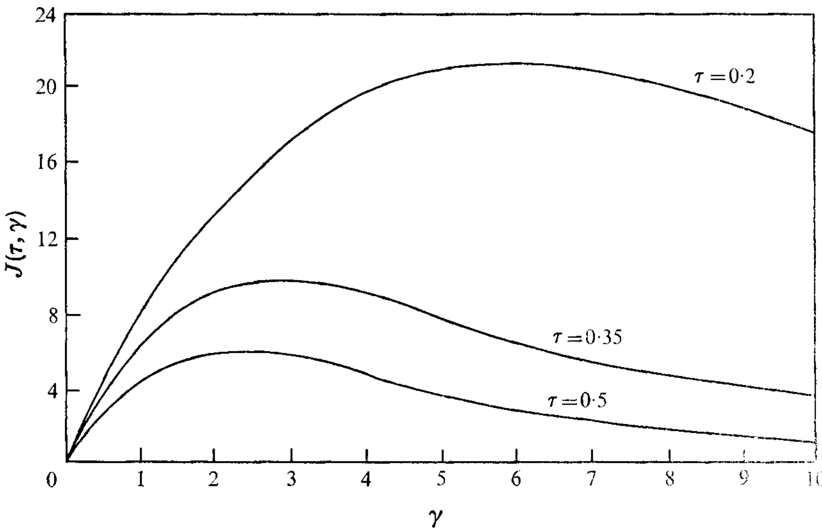

FIGURE 2. Graph of $J(\tau, \gamma)$, defined by (5.3), giving the energy dissipated from the Kelvin boundary wave owing to a coastal perturbation $l(x)$.

seen that there is a particular coastal perturbation which will give rise to maximum energy propagation away from the coast. This maximum occurs in each case when $\gamma\tau \simeq 1{\cdot}2$, corresponding by the definition above to a coastal perturbation length $4/\gamma \simeq 4\tau/1{\cdot}2 \simeq \pi\tau$, i.e. approximately half the Kelvin wavelength. If γ is small, corresponding to a flat coastline, the Kelvin wave is transmitted with little change, much as would be expected. What is perhaps less expected, however, is that the same tendency is predicted for coastlines which are abrupt for γ large. Of course if γ is too large, corresponding to a very abrupt perturbation, one of the basic assumptions of the present theory is violated. However, as indicated previously the theory is valid provided $4/\gamma \geqslant O(1)$, and in consequence the results illustrated in figure 2 are reasonably acceptable for the range of γ taken.

As τ decreases, the propagation of energy away from a given boundary increases, in accordance with other papers on this topic mentioned in § 1. The value of γ for the maximum propagation then also increases as would be reasonably expected, since the boundary will have to be shorter to have the greatest effect on the waves of smaller τ and smaller wavelength.

5.2. *Local amplitude and phases changes*

Munk *et al.* (1970)† give details of amplitude and phase variations in diurnal ($\sigma = 1$) and semi-diurnal ($\sigma = 1{\cdot}97$) tidal constituents detected along the Californian coast. They suggest that these anomalies may be attributable to coastal irregularities. The amplitude of the waves detected were found to decrease exponentially with distance away from the coastline, as of course do the Kelvin waves of the present treatment. It is therefore relevant to inquire here if similar variations can be predicted for a Kelvin wave moving in the same circumstances.

† The author is indebted to a referee for drawing his attention to this paper.

To obtain the local changes undergone by an incident Kelvin wave as it travels along an irregular coastline it is sufficient to consider only the first-order theory. Confining attention to coastal boundary changes only, (3.4) yields on $y = 0$, after putting $m = 0$, $n = -i$, that for a Kelvin wave

$$\bar{\zeta}_1(\alpha, 0) = -\frac{\bar{l}(\alpha+\sigma)(\sigma^2-1)\,\sigma(\alpha+\sigma)}{[\sigma(\alpha^2-\sigma^2+1)^{\frac{1}{2}}+\alpha]}. \tag{5.4}$$

Together with (2.5), (2.12) and (3.1*b*), this yields the coastal wave elevation

$$\zeta(x, 0, t) = e^{i\sigma x - i\omega t}\{1+D\}, \tag{5.5}$$

where
$$D = \frac{\epsilon\sigma}{2\pi}\int_{-\infty}^{\infty} \bar{l}(\alpha+\sigma)\,\frac{[\alpha-\sigma(\alpha^2-\sigma^2+1)^{\frac{1}{2}}]}{\alpha-\sigma}\,e^{-i(\alpha+\sigma)x}\,d\alpha, \tag{5.6}$$

inserting the time dependence $e^{-i\omega t}$. Now

$$\bar{l}(\alpha+\sigma) = \int_{-\infty}^{\infty} l(x,0)\,e^{(\alpha+\sigma)x_0}\,dx_0, \tag{5.7}$$

and on substituting in (5.6) it is possible to justify a change in the order of integration. In so doing the integration of α along the real axis is replaced by integration along cuts parallel to the imaginary α axis which exclude the branch points $\alpha = \pm(\sigma^2-1)^{\frac{1}{2}}$ (dealing only with the case of immediate interest, $\sigma \geqslant 1$).

Thus it can be shown that

$$D = \int_{-\infty}^{\infty} l(x_0)\,G(x-x_0)\,dx_0, \tag{5.8}$$

where

$$G(x-x_0) = \pm\frac{\sigma^2\epsilon}{\omega\pi}\,e^{i[(\sigma^2-1)^{\frac{1}{2}}\pm\sigma]|x_0-x|}\int_0^{\infty}\frac{[s^2-2i(\sigma^2-1)^{\frac{1}{2}}s]^{\frac{1}{2}}}{(\sigma^2-1)^{\frac{1}{2}}\mp\sigma+is}\,e^{-|x_0-x|\,s}\,ds, \tag{5.9}$$

the upper sign applying if $x_0 - x > 0$ and the lower sign if $x_0 - x < 0$. On separation into real and imaginary parts (5.9) can be easily evaluated numerically. It then remains to determine D from (5.8) by a further integration once the particular coastline is specified.

The section of coastline under consideration is illustrated in figure 3(*c*). Its shape includes a number of small-scale irregularities falling outside the scope of the linearized theory. Such irregularities are in fact considered by Miles & Munk (1961) and Buchwald (1971). For the sake of comparison it is assumed here that the average effect of these small-scale changes can be ignored, and that the changes which have a large lengthwise scale can be dealt with by replacing the actual coastline with a smoothed outline, also shown in figure 3(*c*). The perturbation $l(x)$ is taken to be the departure from the straight line (or great circle) coinciding approximately with the 2000 km of reasonably straight Canadian coastline which occurs just north of the coastal section under consideration. The depth at great distances from the coast is $d = 3{\cdot}4$ km. The representative length along the coast is taken to be $(gd)^{\frac{1}{2}}/f = 2600$ km.

The tidal amplitudes determined from observations along the Californian coast by Munk, Snodgrass & Wimbush are shown in figure 3(*a*) and are compared in the cases $\sigma = 1$, $1{\cdot}97$ with values calculated from (5.5) and (5.8). The present

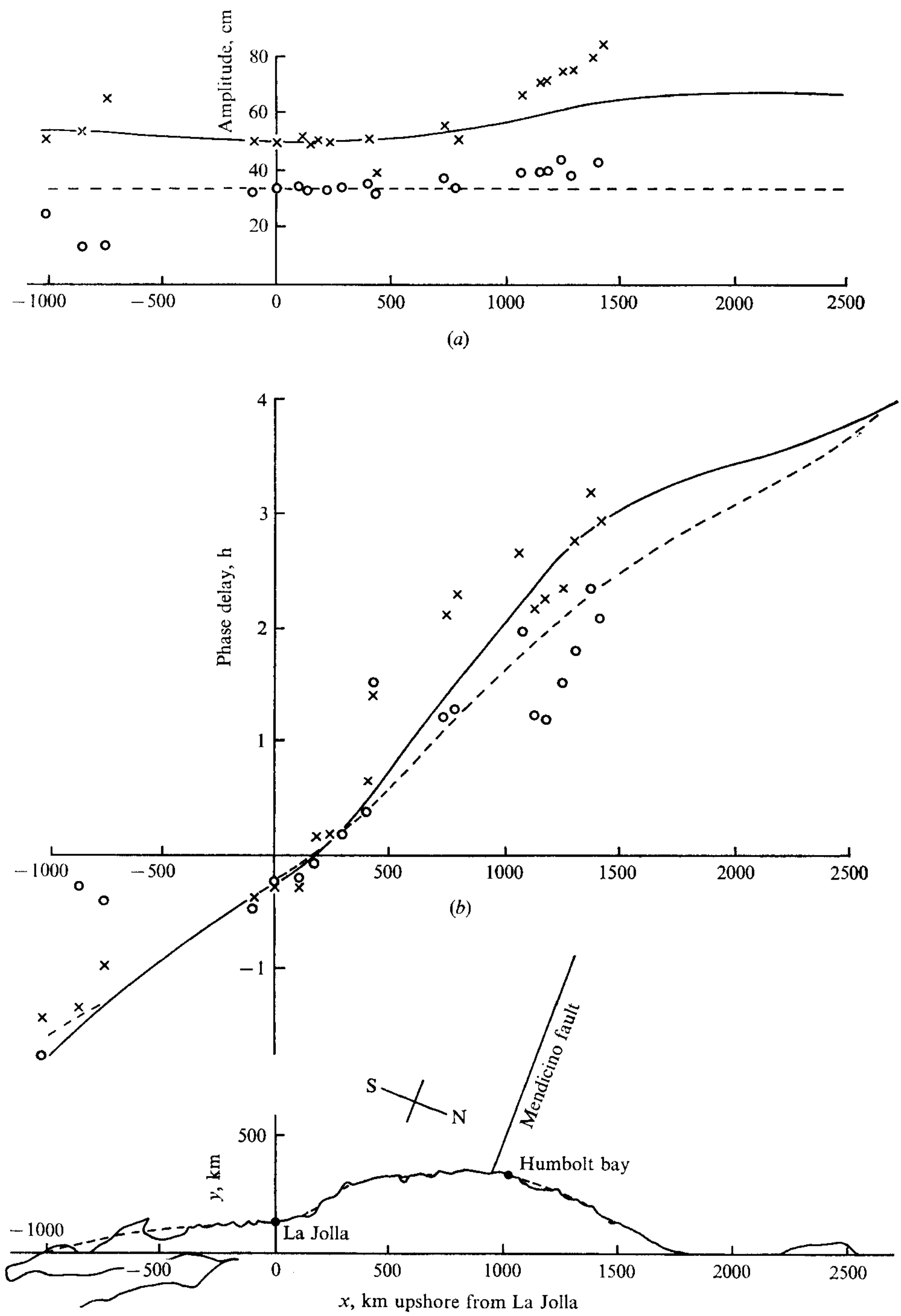

FIGURE 3. (*a*) Comparison of the amplitude changes for the Kelvin waves and the tidal constituents. ----, ○, $\sigma = 1$; ——, ×, $\sigma = 1\cdot97$. (*b*) Phase time delays relative to the datum line for a Kelvin wave propagating northwards with a velocity of 685 km/h. – – –, diurnal wave, $\sigma = 1$; ——, semi-diurnal wave, $\sigma = 1\cdot97$. Phase time delays, made to coincide at La Jolla, of the tidal constituents obtained from observation: ○, $\sigma = 1$; ×, $\sigma = 1\cdot97$. (*c*) Section of Californian coastline under consideration. The smoothed shape $l(x)$ (taking $l(x) > 0$) is also shown relative to the x axis datum line.

theory predicts only relative changes in amplitude and therefore in order to effect a comparison the detected values of amplitude at La Jolla have been incorporated. The corresponding results for phase measured in terms of time delay are shown in figure 3 (*b*). These are superimposed on the time phase change of the straight line datum chosen assuming the Kelvin wave to be moving northwards with speed $(gd)^{\frac{1}{2}} = 685\,\text{km/h}$.

The calculation shows that north of La Jolla results are not sensitive to moderate changes in the datum line. The phase times predicted at La Jolla are $-12{\cdot}5$ min for $\sigma = 1$ and -16 min for $\sigma = 1{\cdot}97$. Since Munk, Snodgrass & Wimbush give their results in terms of phase delays relative to those at La Jolla this is the sense in which their results are shown for comparison in figure 3. Discounting small-scale discrepances, it would appear that the overall trends are similar for the detected anomalies and those which have been calculated by the present theory for a Kelvin wave. In particular the present method certainly yields the right order of phase time delay at Humbolt Bay, i.e. 20 min for $\sigma = 1$ and 50 min for $\sigma = 1{\cdot}97$. In addition it shows that the amplitude varies much more for the $\sigma = 1{\cdot}97$ case than for the $\sigma = 1$ diurnal wave.

The discrepancies south of La Jolla are perhaps not unexpected in view of the protruding coastline in this region and the proximity of the Southern Californian Cape. However, the differences just north of Humbolt Bay are more surprising. It is suspected these are due to the occurrence of the Mendicino fault line, which leaves the coast near this position. However, using the linearized theory starting from (3.4), and allowing for a suitable depth decrease north of Cape Mendicino, revealed only gradual phase changes, more or less equivalent to a slower moving Kelvin wave propagating with the appropriate reduced velocity.

6. Conclusions

A straight coast bounding a sea of constant depth reflects perfectly a system of incident plane waves. The presence of a small perturbation, for example, a coastal change ΔL or a depth change ΔH, leads in general to some of the incident energy being extracted and used to form Kelvin waves propagating along the boundary away from the disturbance. The ratio of the Kelvin wave amplitude so formed in each case to that of the incident wave is of $O[\Delta Lf/(gd)^{\frac{1}{2}}]$ and $O(\Delta H/d)$ respectively. This compares with the corresponding result for a sharp edge boundary (Crease 1956), for which the amplitude ratio is generally $O(1)$.

It is shown that the actual energy extracted from the incident wave system by a nearly straight coastline depends very much on the particular geometry. This is indicated by the complete cancellation of the Kelvin waves in some cases, while in others reinforcement takes place. However, it does appear that when a line of depth changes is present the largest Kelvin wave amplitudes usually form when the incident plane wave crests are nearly parallel to this line.

In the case when the incident wave has much less energy and is itself a Kelvin wave, the proportional amplitude changes at large distance along the coast are $O[\Delta Lf/(gd)^{\frac{1}{2}}]^2$ and $O(\Delta H/d)^2$ for coastal and depth perturbations, respectively. This only occurs if $\sigma > 1$ when energy is diverted away from the boundary in the form of cylindrical Poincaré waves, the greatest effect being attained when a given perturbation has a characteristic length similar to that of the incident wave. For a particular period τ, some combinations of coastal and depth perturbations cause no Poincaré waves to be set up. Such geometries have the interesting dual property of being unable to extract energy from incident plane waves of period τ and form Kelvin waves.

So far these conclusions apply only to changing conditions at large distances from the perturbation. Locally the perturbations cause changes in the amplitude and phase of the wave experienced at the coast even if no net change occurs at large distances. It is shown that along the Californian coastline, provided irregularities of small lengthwise scale are ignored, an incident Kelvin wave will suffer changes in amplitude and phase similar to changes found in tidal wave constituents determined from observations. As the period of the incident wave τ becomes smaller there would appear to be a tendency for amplitude and phase variations caused by a given coastline to increase.

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